Extension of Wirtinger's Calculus in RKH Spaces and the Complex Kernel LMS

A unified framework for complex signal processing in RKHS

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31-08-2010

Outline

1. Signal Processing with Kernels
   - Preliminaries
   - Kernel LMS
2. Wirtinger's Calculus
   - The Complex Case: Wirtinger's Calculus
   - Wirtinger's Calculus in complex RKHS
3. Complex Kernel LMS
   - Formulation
   - Sparsification
   - Experiments

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This procedure is equivalent with a non linear processing in F.

Reproducing Kernel Hilbert Spaces.

Consider a linear class $\mathcal{H}$ of real valued functions f defined on a set X (in particular $\mathcal{H}$ is a Hilbert space), for which there exists a function $\kappa : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ with the following two properties:

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1. For every $x \in \mathcal{X}$, $\kappa(\cdot, x)$ belongs to $\mathcal{H}$.
2. κ has the so called reproducing property, i.e.,

$$f(x) = \langle f, \kappa(\cdot, x)\rangle_{\mathcal{H}}, \text{ for all } f \in \mathcal{H},\ x \in \mathcal{X}. \tag{1}$$

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then the inner product in $\mathcal{H}$ is given as a function computed on $\mathcal{X}$:

$$\kappa(x, y) = \langle \kappa(\cdot, y), \kappa(\cdot, x) \rangle_{\mathcal{H}}$$ kernel trick

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- Different types of nonlinearities can be treated in a unified way.

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  - Develop the Algorithm in $\mathcal{X}$.
  - Express it, **if possible**, in inner products.
  - Replace inner products with kernel evaluations according to the kernel trick.
- Work directly in the RKHS, assuming that the data have been mapped and live in the RKHS $\mathcal{H}$, i.e.,

$$\mathcal{X} \ni \boldsymbol{x} \rightarrow \Phi(\boldsymbol{x}) := \kappa(\cdot, \boldsymbol{x}) \in \mathcal{H}.$$

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Development of a unified framework for complex valued signal processing in RKHS.

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- Using the derivative of the cost, the gradient descent update rule becomes: $\boldsymbol{w}(n) = \boldsymbol{w}(n-1) + \mu e(n)\boldsymbol{x}(n)$.
- The desired output becomes
$\hat{d}(n) = \boldsymbol{w}(n-1)^T\boldsymbol{x}(n) = \mu \sum_{k=1}^{n-1} e(k)\,\boldsymbol{x}(k)^T\boldsymbol{x}(n)$.

Kernel LMS

- In Kernel LMS, firstly we transform the input space to a RKHS $\mathcal{H}$ to obtain the sequence: $(\Phi(\boldsymbol{x}(1)), d(1)), (\Phi(\boldsymbol{x}(2)), d(2)), \ldots, (\Phi(\boldsymbol{x}(N)), d(N))$.

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- We apply the LMS procedure to the sequence of examples minimizing the cost function $E[|d(n) - \langle \Phi(\boldsymbol{x}(n)), \boldsymbol{w} \rangle_{\mathcal{H}}|^2]$, where now $\boldsymbol{w} \in \mathcal{H}$.

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- The filter output of the KLMS is: $\hat{d}(n) = \langle \boldsymbol{x}(n), \boldsymbol{w}(n-1) \rangle_{\mathcal{H}} = \mu \sum_{k=1}^{n-1} e(k)\, \kappa(\boldsymbol{x}(k), \boldsymbol{x}(n))$.

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An operator $T : \mathcal{H} \to F$ is said to be Fréchet differentiable at f_0, if there exists $u \in \mathcal{H}$ such that the limit

$$\lim_{\|h\|_{\mathcal{H}} \to 0} \frac{T(f_0 + h) - T(f_0) - \langle u, h \rangle_{\mathcal{H}}}{\|h\|_{\mathcal{H}}} = 0.$$

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We say that f is differentiable at x iff the following limit exists:

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$$\lim_{\|h\|_{\mathcal{H}}\to 0}\frac{T(f_0+h)-T(f_0)-\langle u,h\rangle_{\mathcal{H}}}{\|h\|_{\mathcal{H}}}=0.$$

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Complex and real derivatives

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We will say that f is differentiable in the complex sense at c (or that it has complex derivative at c), iff the limit

$$\lim_{z \to c} \frac{f(z) - f(c)}{z - c}$$

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- In complex signal processing we often encounter functions (e.g., the cost functions, which are defined in $\mathbb{R}$) that ARE NOT complex differentiable.
- Example: $f(z) = |z|^2 = zz^*$.
- In these cases one has to express the cost function in terms of its real part f_r and its imaginary part f_i, and use real derivation with respect to f_r, f_i.

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- These rules bear a great resemblance to the rules of the standard complex derivative.

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- The conjugate $\mathbb{R}$-derivative:
  $$\frac{\partial f}{\partial z^*} = \frac{1}{2}\left(\frac{\partial f_r}{\partial x} - \frac{\partial f_i}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_i}{\partial x} + \frac{\partial f_r}{\partial y}\right).$$

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- The main rules and principles are similar.
- In order to extend it to a complex RKHS (where the dimensionality can be infinite), we need to employ the notion of Fréchet differentiability.

Wirtinger's derivatives in RKHS

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Any gradient descent based algorithm minimizing a real valued operator $\boldsymbol{T}(\boldsymbol{f})$ is based on the update scheme:

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Remark: We have used $\boldsymbol{f}$ in place of $\boldsymbol{w}$ (used before) to stress the fact that the RKHS $\mathbb{H}$ can be of infinite dimension.

Outline

Mapping to the complex RKHS

- Consider the sequence of examples $(\boldsymbol{z}(1), d(1)), (\boldsymbol{z}(2), d(2)), \dots (\boldsymbol{z}(N), d(N))$, where $d(n) \in \mathbb{C}$ and $\boldsymbol{z}(n) \in \mathbb{C}^\nu$

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- Let $\boldsymbol{z}(n) = \boldsymbol{x}(n) + i\boldsymbol{y}(n)$, $\boldsymbol{x}(n), \boldsymbol{y}(n) \in \mathbb{R}^{\nu}$.
- We map the points $\boldsymbol{z}(n)$ to the complex RKHS $\mathbb{H}$ an appropriate complex mapping $\boldsymbol{\Phi}$.

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- $\boldsymbol{\Phi}$ can be the result of complexifing real kernels:

$$\begin{aligned}\boldsymbol{\Phi}(\boldsymbol{z}(n)) &= \Phi(\boldsymbol{z}(n)) + i\Phi(\boldsymbol{z}(n)) \\ &= \kappa\left((\boldsymbol{x}(n), \boldsymbol{y}(n))^T, \cdot\right) + i \cdot \kappa\left((\boldsymbol{x}(n), \boldsymbol{y}(n))^T, \cdot\right),\end{aligned}$$

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- The latter choice has been used in this work, using the real gaussian kernel. This is because the behavior of such kernels is well understood in SP applications.
- Note that when complexified real kernels are employed, the complex kernel LMS **CANNOT** be derived by applying the standard kernel trick on the complex LMS (details in the paper).

Complex Kernel LMS

- We apply the complex LMS to the transformed data: $(\boldsymbol{\Phi}(\boldsymbol{z}(1)), d(1)), (\boldsymbol{\Phi}(\boldsymbol{z}(2)), d(2)), \ldots (\boldsymbol{\Phi}(\boldsymbol{z}(N)), d(N))$.

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- The objective of CKLMS is to minimize

$$E[|e(n)|^2] = E[|d(n) - \langle \mathbf{\Phi}(\boldsymbol{z}(n)), \boldsymbol{f} \rangle_{\mathbb{H}}^2],$$

at each instance n.

Complex Kernel LMS

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$$\boldsymbol{f}(n) = \boldsymbol{f}(n-1) + \mu e(n)^* \cdot \boldsymbol{\Phi}(\boldsymbol{z}(n)),$$

where $\boldsymbol{f}(n)$ denotes the estimate at iteration n.

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=& \boldsymbol{f}(n-2) + \mu e(n-1)^* \boldsymbol{\Phi}(\boldsymbol{z}(n-1)) \\
& + \mu e(n)^* \boldsymbol{\Phi}(\boldsymbol{z}(n)) \\
=& \sum_{k=1}^{n} e(k)^* \boldsymbol{\Phi}(\boldsymbol{z}(k)).
\end{aligned}$$

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The filter output at iteration *n* becomes:

$$
\begin{aligned}
\hat{d}(n) =& \langle \boldsymbol{\Phi}(\boldsymbol{z}(n)), \boldsymbol{w}(n-1) \rangle_{\mathbb{H}} \\
=& \mu \sum_{k=1}^{n-1} e(k) \langle \boldsymbol{\Phi}(\boldsymbol{z}(n)), \boldsymbol{\Phi}(\boldsymbol{z}(k)) \rangle_{\mathbb{H}} \\
=& 2\mu \sum_{k=1}^{n-1} e(k) \kappa(\boldsymbol{z}(n), \boldsymbol{z}(k)) \\
=& 2\mu \sum_{k=1}^{n-1} \Re[e(n)] \kappa(\boldsymbol{z}(n), \boldsymbol{z}(k)) + 2\mu \cdot i \sum_{k=1}^{n-1} \Im[e(n)] \kappa(\boldsymbol{z}(n), \boldsymbol{z}(k)),
\end{aligned}
$$

Outline

Sparsification

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- Results: Increasing memory and computational requirements.
- A sparse solution is needed.
- Any sparsification algorithm can be employed. Details are given in the paper.

Outline

Experiments

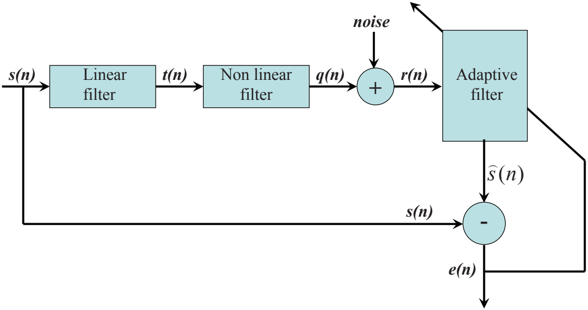

Figure: The equalization problem.

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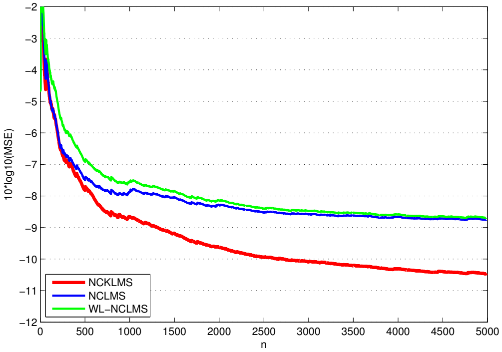
  2. highly non-circular if ρ approaches 0 or 1.

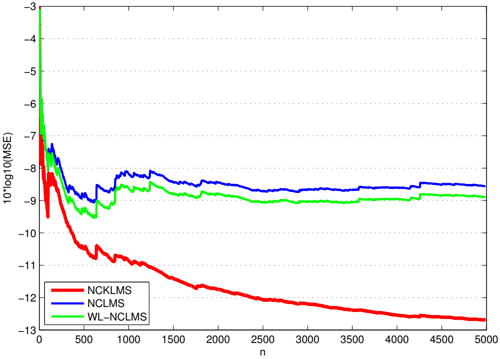
Circular Data

Learning curves for KNCLMS ($\mu = 1/2$), NCLMS ($\mu = 1/16$) and WL-NCLMS ($\mu = 1/16$) (filter length $L = 5$, delay $D = 2$) in the nonlinear channel equalization, for the **circular** input case.



Non Circular Data

Learning curves for KNCLMS ($\mu = 1/2$), NCLMS ($\mu = 1/16$) and WL-NCLMS ($\mu = 1/16$) (filter length $L = 5$, delay $D = 2$) in the nonlinear channel equalization, for the **non-circular** input case ($\rho = 0.1$).



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3. The development of the Complex Kernel LMS algorithm as a particular example.
    - Experiments verify that CKLMS gives significantly better results compared to CLMS and WL-CLMS for nonlinear channels.